Thruster Placement, Transformations, and Differentiable Evaluation for 6-DoF Wrench Authority

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1 Introduction

We consider N body-fixed thrusters mounted on or near a reference sphere of radius ρ. Each thruster produces a force along its own local axis, and we seek to (a) map these forces to a global wrench about the origin and (b) optimize placement/orientation for strong, well-conditioned 6-DoF authority. This document standardizes frames, derives the required transformations, builds the allocation matrix, and proposes a differentiable evaluation function for gradient-based design.

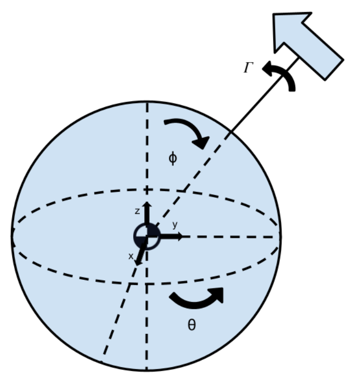

Figure 1: non-Gimbaled Thruster ROV Convention

2 Frames, Coordinates, and Conventions

Base frame. The global Cartesian frame $\{B\}$ has orthonormal axes (x, y, z) about the origin.

Spherical coordinates. We adopt (ρ, θ, ϕ) with azimuth θ about $+z$ from $+x$, and polar angle ϕ measured from $+z$ (north pole). The radial unit vector and tangent basis at (θ, ϕ) are

$$\boldsymbol{e}_r = \begin{bmatrix} \mathsf{c}_\theta \\ \mathsf{s}_\theta \\ c_\phi \end{bmatrix}, \quad \boldsymbol{e}_\theta = \begin{bmatrix} -s_\theta \\ c_\theta \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_\phi = \begin{bmatrix} c_\phi c_\theta \\ c_\phi s_\theta \\ - \end{bmatrix}, \qquad (\boldsymbol{e}_\theta \times \boldsymbol{e}_\phi = \boldsymbol{e}_r). \tag{1}$$

3 Cartesian → Spherical: Coordinates and Basis

Point coordinates (nonlinear). For a Cartesian point $\boldsymbol{p}_B = (x, y, z)^\top$,

$$\rho = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \operatorname{atan2}(y, x), \qquad \phi = \arccos\left(\frac{z}{\rho}\right) \ (\rho > 0). \tag{2}$$

Basis change for vectors (linear). At (θ, ϕ),

$$\begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix} = \underbrace{\begin{bmatrix} \mathsf{c}_\theta & \mathsf{s}_\theta & c_\phi \\ -s_\theta & c_\theta & 0 \\ c_\phi c_\theta & c_\phi s_\theta & - \end{bmatrix}}_{\boldsymbol{C}_{B\to\mathrm{sph}}(\theta,\phi)} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}, \qquad \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{e}_r & \boldsymbol{e}_\theta & \boldsymbol{e}_\phi \end{bmatrix}}_{\boldsymbol{C}_{\mathrm{sph}\to B}} \begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix}. \tag{3}$$

4 Surface and Thruster Transforms

Define the *surface frame* $\{S\}$ at position $\rho\,\boldsymbol{e}_r$ with axes $(x_s, y_s, z_s) = (\boldsymbol{e}_\theta, \boldsymbol{e}_\phi, \boldsymbol{e}_r)$. The rotation and translation from $\{B\}$ to $\{S\}$ are

$$\boldsymbol{R}_{BS} = \begin{bmatrix} \boldsymbol{e}_\theta & \boldsymbol{e}_\phi & \boldsymbol{e}_r \end{bmatrix} = \begin{bmatrix} -s_\theta & c_\phi c_\theta & \mathsf{c}_\theta \\ c_\theta & c_\phi s_\theta & \mathsf{s}_\theta \\ 0 & - & c_\phi \end{bmatrix}, \qquad \boldsymbol{p}_{BS} = \rho\,\boldsymbol{e}_r. \tag{4}$$

The homogeneous transform is

$$\boldsymbol{T}_{BS} = \begin{bmatrix} \boldsymbol{R}_{BS} & \boldsymbol{p}_{BS} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}. \tag{5}$$

Let the *thruster frame* $\{T\}$ be obtained by a rotation γ about $z_s = \boldsymbol{e}_r$ within the tangent plane, then a standoff d along $+z_s$:

$$\boldsymbol{R}_{ST}(\gamma) = \begin{bmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{p}_{ST} = d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\{S\}}. \tag{6}$$

The cumulative transform is

$$\boldsymbol{T}_{BT} = \boldsymbol{T}_{BS}\,\boldsymbol{T}_{ST}, \qquad \boldsymbol{R}_{BT} = \boldsymbol{R}_{BS}\boldsymbol{R}_{ST}, \qquad \boldsymbol{p}_{BT} = \boldsymbol{p}_{BS} + \boldsymbol{R}_{BS}\boldsymbol{p}_{ST} = (\rho + d)\,\boldsymbol{e}_r. \tag{7}$$

5 Force and Wrench Mapping

A pure force is rotated but not translated:

$$\boldsymbol{F}_B = \underbrace{\boldsymbol{R}_{BT}}_{\boldsymbol{M}_F(\theta,\phi,\gamma)} \boldsymbol{F}_T. \tag{8}$$

Carrying out the product $\boldsymbol{R}_{BS}\boldsymbol{R}_{ST}$ yields

$$\boldsymbol{M}_F(\theta,\phi,\gamma) = \begin{bmatrix} -c_\gamma\,s_\theta + s_\gamma\,c_\phi\,c_\theta & s_\gamma\,s_\theta + c_\gamma\,c_\phi\,c_\theta & \mathsf{c}_\theta \\ c_\gamma\,c_\theta + s_\gamma\,c_\phi\,s_\theta & -s_\gamma\,c_\theta + c_\gamma\,c_\phi\,s_\theta & \mathsf{s}_\theta \\ -s_\gamma & -c_\gamma & c_\phi \end{bmatrix}. \tag{9}$$

If the thruster produces a wrench $(\boldsymbol{F}_T, \boldsymbol{\tau}_T)$ at $\{T\}$, the base wrench is

$$\begin{bmatrix} \boldsymbol{F}_B \\ \boldsymbol{\tau}_B \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{R}_{BT} & \boldsymbol{0} \\ \boldsymbol{p}_{BT} mu \boldsymbol{R}_{BT} & \boldsymbol{R}_{BT} \end{bmatrix}}_{\boldsymbol{A}^*(\rho,\theta,\phi,\gamma,d)} \begin{bmatrix} \boldsymbol{F}_T \\ \boldsymbol{\tau}_T \end{bmatrix}, \tag{10}$$

where $\boldsymbol{p} muisthecross-productmatrix.$

6 Thruster Wrench Basis ($6 \times N$ Allocation)

For thruster i with parameters $(\rho_i, \theta_i, \phi_i, \gamma_i, d_i)$, position $\boldsymbol{r}_i = (\rho_i + d_i)\,\boldsymbol{e}_{r_i}$ and unit tangential direction $\hat{\boldsymbol{f}}_i = \cos\gamma_i\,\boldsymbol{e}_{\theta_i} + \sin\gamma_i\,\boldsymbol{e}_{\phi_i}$, define the unit wrench

$$\boldsymbol{b}_i = \begin{bmatrix} \hat{\boldsymbol{f}}_i \\ \boldsymbol{r}_i \times \hat{\boldsymbol{f}}_i \end{bmatrix} \in \mathbb{R}^6. \tag{11}$$

Stacking columns yields the allocation matrix

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_N \end{bmatrix} \in \mathbb{R}^{6\times N}, \qquad \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{\tau} \end{bmatrix} = \boldsymbol{B}\,\boldsymbol{u}, \tag{12}$$

where $\boldsymbol{u} \in \mathbb{R}^N$ are thrust magnitudes (signed if bidirectional).

7 Differentiable Evaluation of the Basis

For forces (N) and torques (N m) we adopt a balancing length $R_{\text{char}} > 0$ and define

$$\boldsymbol{W} = \operatorname{diag}\Big(1, 1, 1, \tfrac{1}{R_{\text{char}}}, \tfrac{1}{R_{\text{char}}}, \tfrac{1}{R_{\text{char}}}\Big), \qquad \widetilde{\boldsymbol{B}} = \boldsymbol{W}\,\boldsymbol{B}, \qquad \boldsymbol{M} = \widetilde{\boldsymbol{B}}\,\widetilde{\boldsymbol{B}}^\top. \tag{13}$$

With a small $\varepsilon > 0$, we combine three smooth terms into a maximization objective:

$$J_{\text{vol}}(\boldsymbol{B}) = \log\det\big(\boldsymbol{M} + \varepsilon \boldsymbol{I}_6\big), \tag{14}$$

$$P_{\text{iso}}(\boldsymbol{B}) = \Big\| \boldsymbol{M} - \tfrac{\operatorname{tr}(\boldsymbol{M})}{6} \boldsymbol{I}_6 \Big\|_F^2, \tag{15}$$

$$P_{\text{coh}}(\boldsymbol{B}) = \sum_{i \neq j} \left(\frac{\widetilde{\boldsymbol{b}}_i^\top \widetilde{\boldsymbol{b}}_j}{\|\widetilde{\boldsymbol{b}}_i\|\,\|\widetilde{\boldsymbol{b}}_j\|} \right)^2, \tag{16}$$

and the combined score

$$\boxed{J(\boldsymbol{B}) = \alpha\,J_{\text{vol}}(\boldsymbol{B}) - \beta\,P_{\text{iso}}(\boldsymbol{B}) - \gamma\,P_{\text{coh}}(\boldsymbol{B}),} \tag{17}$$

with weights $\alpha, \beta, \gamma > 0$. The gradient of J_{vol} admits the closed form

$$\frac{\partial J_{\text{vol}}}{\partial \widetilde{\boldsymbol{B}}} = 2\big(\boldsymbol{M} + \varepsilon \boldsymbol{I}_6\big)^{-1} \widetilde{\boldsymbol{B}}, \qquad \frac{\partial J_{\text{vol}}}{\partial \boldsymbol{B}} = \boldsymbol{W}^\top \frac{\partial J_{\text{vol}}}{\partial \widetilde{\boldsymbol{B}}}. \tag{18}$$

8 Optimization Statement and Practical Constraints

We optimize angular parameters $\{\phi_i, \theta_i, \gamma_i\}_{i=1}^{N}$ (and optionally d_i) subject to bounds and collision/clearance constraints:

$$\max_{\{\phi_i,\theta_i,\gamma_i\}} \quad J(\boldsymbol{B}(\{\phi_i, \theta_i, \gamma_i\})) \tag{19}$$

$$\text{s.t.} \quad \phi_i \in [0, \pi],\ \theta_i \in [0, 2\pi),\ \gamma_i \in [0, 2\pi), \tag{20}$$

$$d_i \in [d_{\min}, d_{\max}],\ \text{clearances, FOV, wiring, and mechanical limits.} \tag{21}$$

Choose R_{char} as a representative arm length, e.g. mean $(\rho_i + d_i)$.

9 Conclusion

This narrative consolidates geometry, transforms, wrench mapping, the allocation matrix, and a differentiable objective suitable for gradient-based thruster placement. It is intended to be self-contained and implementation-ready.