3/23/2025 **To:** Dr. Heather Lewis

From: Ian Wilhite, *ICW*

Subject: Experimental natural frequency analysis

Abstract

In many mechanical systems that undergo typical harmonic excitation, vibrations are inevitable. Engines, buildings, and generators all undergo some type of vibration and oscillation and must determine the extent that harmonic excitation may impact a system, and therefore engineers need a robust methodology to numerically determine and experimentally verify the natural frequency of their system. In this experiment we seek to predict and experimentally verify the natural frequency of vibration for the mechanical systems provided. The beam's dimensions were measured, and the Virtual Instrument (VI) in LabVIEW captured displacement traces under different loads [1]. Calculated damping ratios, logarithmic decrements, and natural frequencies were compared against theoretical values derived from beam theory. Experimental natural frequencies for both deflection magnitudes averaged 93.79 rad/s, while the theoretical value was 167.95 rad/s. A 95% confidence interval and two-sample t-test confirmed consistency across both trials. Furthermore, linear regression was used to calibrate strain measurements and predict an unknown weight, validating the effectiveness of the calibration process. Sources of error in the collected values of the traces include calibration uncertainty, temperature fluctuations, and sensor noise. Results deviated from theoretical predictions, likely due to accumulation of sources of error in measured values and sensor calibration for experimental values. This reiterates the importance of analysis on the vibrations of harmonically excited mechanical systems and the impacts that those vibrations can have on the performance and life cycle of those systems. Future work could include additional trials to mitigate the impact of inconsistencies in trials and analysis on additional peaks pending on the noise in the sensors used to measure amplitude.

Introduction

The experiment performed is designed to analyze the effects of the vibrations in a cantilevered beam. These vibrations, if not properly accounted for, can lead to fatigue, noise, inefficiency, or catastrophic failure in components ranging from engine mounts and turbine blades to bridges and aerospace structures. Understanding the methodologies to predict the natural frequency is essential to many oscillating systems which receive harmonic excitation including engines, heavy machinery, and large structures [3]. In this experiment, the cantilever beam's dimensions were measured to calculate the equivalent system mass and area moment of inertia across the axis of deflection. These processes rely on the generated trend between collected datapoints, which

The Virtual Instrument (VI) in LabVIEW was utilized to capture the displacement of the beam under loading conditions [1]. The experimentally found traces were used to calculate the logarithmic decrement, damping ratio, and natural frequency for the vibrations under small and large deflection. The theoretical natural frequency was determined using beam theory and compared to the experimental values obtained from the displacement traces. A 95% confidence interval was constructed to find the bounds of the natural frequency. Similarly, various masses were applied, and the deflection of the beam was determined. Then a trendline was created and used to predict an unknown mass. This process highlights the validity of the sensors and data collection processes, and their ability to verify the calibration and determine the predictive value in the data collected.

Procedure

First the properties of the beam should be found including length, width, and thickness of the cantilever beam. The material should be noted so that the material properties can be referenced later.

$$M_e[kg] = 0.23 * M_{beam}[kg] + m_{applied}[kg] \tag{1}$$

The equivalent mass can be found using Equation (1), which requires the mass of the beam and the mass applied to the end of the beam. The area moment of inertia for the beam can be found applying Equation (2) using the base width of the beam and the height of the beam.

$$I_{area}[m^4] = \frac{1}{12} b[m] * (h[m])^3 \tag{2}$$

Using the known properties of the beam, we can apply beam theory to determine the natural frequency of oscillation using Equation (3).

$$\omega_n \left[\frac{rad}{sec}\right] = \sqrt{\frac{3EI}{M_e l^3}} \left[\frac{rad}{sec}\right] \tag{3}$$

Where omega is the undamped natural frequency of the beam in unit radians per second, E is Young's modulus of the beam material in unit pascal, I is the area moment of inertia of the beam cross-section in unit meter to the fourth, Me is the equivalent system mass in kilograms, which accounts for the effective mass of the cantilever beam in vibration, and l is the length of the bar in meters.

Using the known masses, the gauge can be calibrated to accurately measure the displacement of the beam. For the first trial, we displaced the beam a small amount and recorded the displacement over time for the vibration until the displacement amplitude decayed to equilibrium. For the second trial, we displaced the beam a large amount and similarly recorded the displacement over time for the vibration of the beam. From the data, we extracted the peak amplitudes and the time at which they occurred, then applied Equation (4) to determine the logarithmic decrement of the system.

$$\delta = \frac{1}{n}\ln\left(\frac{x_n}{x_{n-1}}\right) \quad (4)$$

The damping ratio could then be calculated using the logarithmic decrement between each pair of peaks using Equation (5).

$$\xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \quad (5)$$

In order to construct a confidence interval for each parameter, the t-score must be identified for the scenario. To construct a 95% confidence interval, we identify that for six measured peaks, there are five found logarithmic decrement values, five damping ratios, and five natural frequencies calculated, therefore, to construct a confidence interval, we identify that there are four degrees of freedom and an alpha value of 0.05. We can then use Equation (6) to determine the confidence intervals for the natural frequency of the small and large excitations.

$$CI = \bar{x} \pm t\frac{s}{\sqrt{n}} \quad (6)$$

Using the damped natural frequency and the known eta value, the natural frequency can be experimentally determined for each of the trials known. The relationship between damped and undamped natural frequency is given traditionally by Equation (7).

$$\omega_d = \omega_n\sqrt{1 - \xi^2} \quad (7)$$

Which can be rearranged into a form more useful for analysis as seen in Equation (8).

$$\omega_n = \frac{\omega_d}{\sqrt{1 - \xi^2}} \quad (8)$$

Using the relationship provided for the natural frequency in Equation (8), the propagation of uncertainty can be applied to determine the uncertainty of the experimental natural frequency using Equation (9) [2][4].

$$\delta\omega_n = \sqrt{\left(\delta\omega_d * \frac{\partial\omega_n}{\partial\omega_d}\right)^2 + \left(\delta\xi\,\frac{\partial\omega_n}{\partial\xi}\right)^2} = \sqrt{\left(\delta\omega_d * \frac{1}{\sqrt{1 - \xi^2}}\right)^2 + \left(\delta\xi * \omega_d\frac{-\xi}{(1 - \xi^2)^{-\frac{3}{2}}}\right)^2} \quad (9)$$

In the second part of the experiment, strain measurements were taken as known weights were incrementally applied to the cantilever beam. Using the “Lab04-PartB.vi” file in LabVIEW, the strain (in Ohms) was recorded for masses of 0g, 50g, 100g, 120g, 150g, and 200g. After all known weights were measured, an unknown mass was applied and its strain recorded. These data points were used to perform a linear regression in Excel, enabling the prediction of the unknown weight.

Results and Discussion

The primary goal of the first part of this experiment was to determine the natural frequency of a cantilever beam and compare theoretical predictions with experimental measurements. The data obtained from small and large displacement traces were utilized to calculate logarithmic decrement, damping ratio, and natural frequency.

Table 1: Material properties for the cantilever beam.

Characteristic	**Value**	**Unit**
Material	Steel	n/a
Width	51.17	mm
Thickness	1.56	mm
Length	22.3	cm
Moment of Inertia	1.619*10^-11	m^4
Density	7.85	g/cm^3
Youngs Modulus	207*10^9	Pa
Mass	139.7	g

As shown in Table 1, the beam was made of steel with a width of 51.2 mm, a thickness of 1.56 mm, and a length of 22.3 cm. Using beam theory and Equation (3), the theoretical natural frequency was calculated as167.95 rad/s.

Table 2 and Table 3 summarize the extracted peak amplitudes, time intervals, and calculated parameters from the vibration traces.

Using the experimentally determined damped frequency and damping ratio, the natural frequency can be experimentally determined for each trial, which is displayed in Table 2.

Table 2: Trace natural frequencies and confidence intervals

Trace	**Natural Frequency (rad/s)**	**Uncertainty (rad/s)**	**Confidence Interval**
Small displacement	93.79	1.229	(92.56, 95.02)
Large displacement	93.79	1.229	(92.56, 95.02)

From Table 1, the mean natural frequency for the small displacement trace was 93.79 rad/s with a standard deviation of 0.99 rad/s. The mean damping ratio was calculated as 0.0035, and the logarithmic decrement was found to be 0.0193.

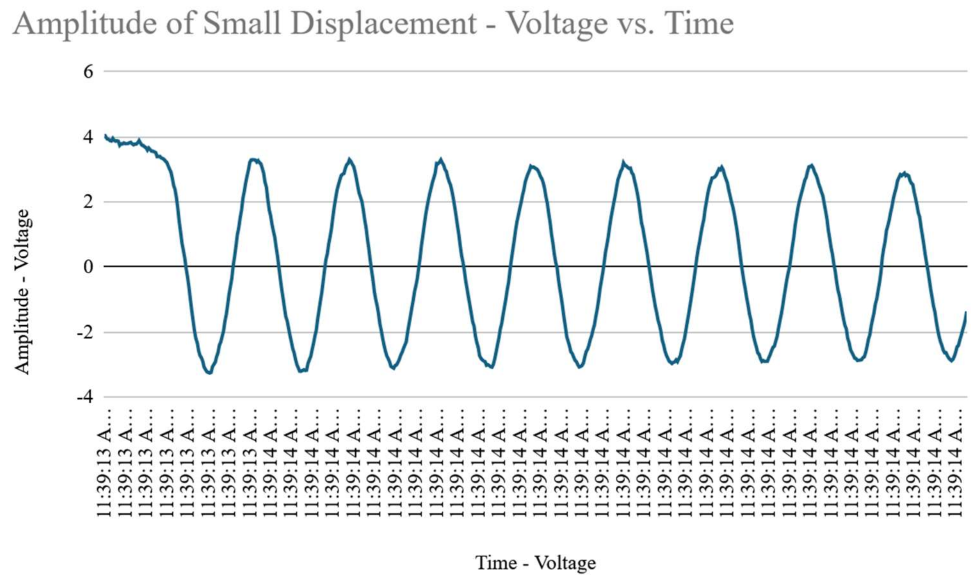

Figure 1: Amplitude of small excitation

For the large displacement trace, the mean natural frequency remained 93.79 rad/sec with a standard deviation of 0.99 rad/sec. The damping ratio and logarithmic decrement were calculated to be 0.0031 and 0.0193, respectively.

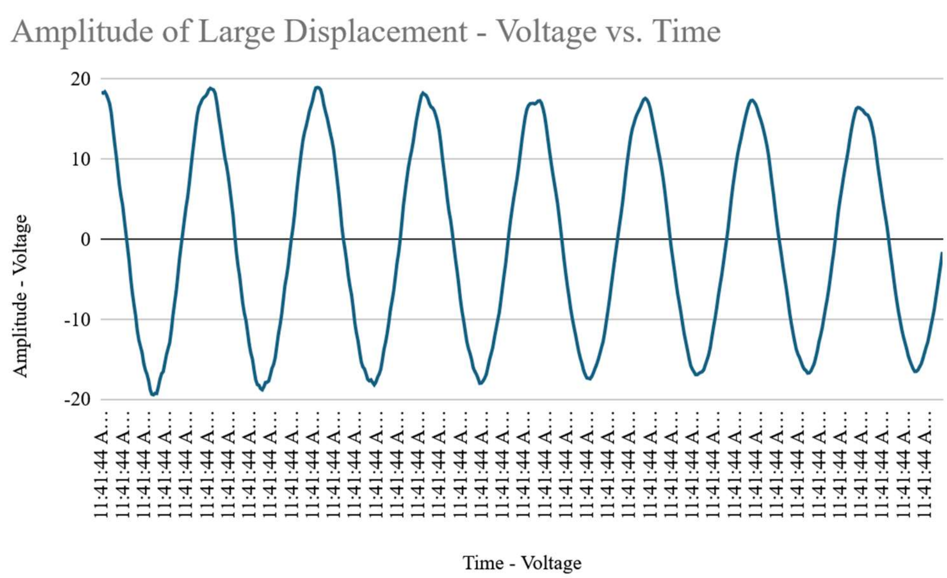

Figure 2: Amplitude of large excitation

A two-sample t-test was performed to determine whether the natural frequencies obtained from the small and large displacement traces were statistically different. The test yielded a t-value of 0.0595, which was less than the critical value of 2.306 for a 95% confidence level and 8 degrees of freedom. Therefore, we fail to reject the null hypothesis, indicating that the two frequencies are statistically equivalent.

The theoretical natural frequency of 167.95 rad/sec deviated significantly from the experimentally determined value of 93.79 rad/sec for both small and large displacements. This difference indicates many sources of error.

After applying the masses to the beam, the resistance values were determined for various masses applied and presented in Table.

Table 3: Summary of found values.

Mass (grams)	Resistivity (Ohms)
0	-0.00370511
50	-0.00367987
100	-0.00365754
120	-0.00365266
150	-0.00363531
200	-0.00361276

The collected data from Table 3 was then plotted in Figure 3 and a trendline plotted.

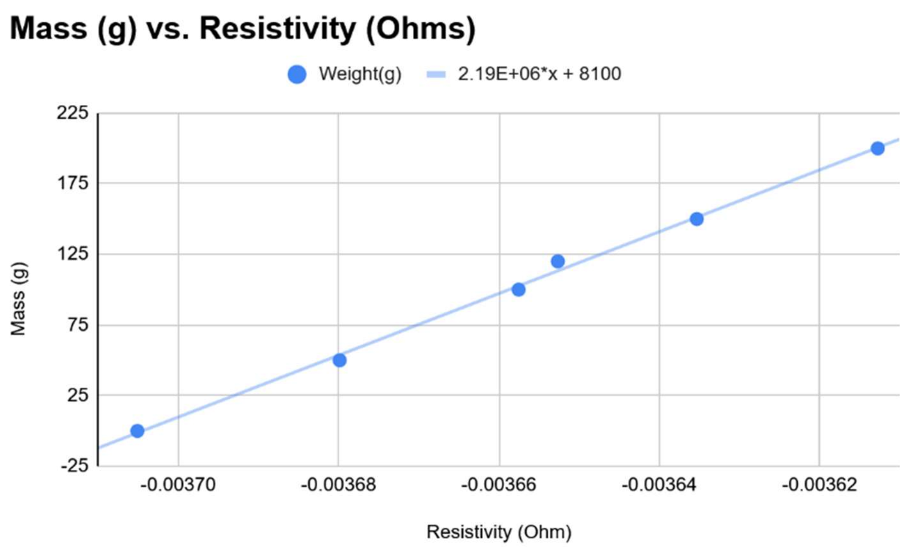

Figure 3: Plotted experimental mass over resistivity

The strain measurements under known weights produced a highly linear relationship (R^2 = 0.996) shown in Equation (10), validating the strain gauge's consistency in the static loading scenario. Using the linear regression model, the unknown mass was predicted with high accuracy, demonstrating the practical utility of the gauge system in low-frequency or static load measurements.

$$W[g] = 2.19 * 10^6 * R + 8100 \tag{10}$$

The unknown strain measurement of -0.00357253 ohms was input into the regression equation, yielding a predicted mass of 276.2 grams.

There were likely errors in the calibration of the strain gauge, which could have introduced a small offset to the measurements or a linear scaling factor into the amplitude but likely not a major effect on the time of the peaks. Minor error could be attributed to temperature fluctuations, material defects, or beam bending asymmetry. Sensor noise during the measurement of the amplitudes likely played a significant role in the noise of the logarithmic decrement which propagated into the damping ratio and the calculated undamped natural frequency. The sensor noise could be identified in the "capping" of the peaks of the displacement graph as well as the general increase in noise at the maximum displacement due to hysteretic effects potentially acting to smooth the amplitude data while the beam was in motion. These effects can be observed in Figures 1 and 2.

Other sources of error could include material defects or other vibrations in the testing assembly supporting the beam being analyzed.

A 95% confidence interval was constructed for both displacement traces to assess the variability in natural frequency. Using Equation (5), the confidence interval was determined and presented in Equations (6), and (7).

Conclusion/Summary

After performing a two-sample t-test, it can be concluded that there is a statistically significant difference between the damped natural frequency and the undamped natural frequency.

The objective of this experiment was to analyze the vibrational behavior of a cantilever beam and compare the theoretical and experimental natural frequencies. Using beam theory, the theoretical natural frequency was calculated to be 167.95 rad/sec, while the experimentally determined values for small and large displacement traces were approximately 93.79 rad/sec. The results demonstrated a significant deviation between theoretical and experimental frequencies. These results indicate major sources of error in the experimental procedure applied or the methodologies presented.

A 95% confidence interval was constructed for both traces, indicating that the experimental results were consistent across different displacement magnitudes. Additionally, a two-sample t-test confirmed that there was no statistically significant difference between the small and large displacement traces, reinforcing the reliability of the measured values.

The second half of the experiment demonstrated the practical use of strain gauge calibration through linear regression. The unknown mass was predicted with high accuracy, validating the reliability of the calibration procedure and the strain gauge system. Together, these two experimental methods showcase the importance of both frequency analysis and accurate calibration in mechanical measurement systems.

Despite the discrepancies observed, the experiment provided valuable insights into the importance of accounting for measurement errors and environmental factors when analyzing vibrational systems [3]. Future works could include minimizing sensor noise, or refining calibration methods. Overall, this experiment emphasized the need for additional work in understanding and predicting the natural frequencies of mechanical systems.

References

[1] Gilman, J., and Glasofer, J., 2003, "Introduction into LabVIEW Programming, MEEN 260 Laboratory Manual," Texas A&M University.

[2] Beckwith, T. G., Marangoni, R. D., and Lienhard, J. H., 2007, *Mechanical Measurements*, 6th ed., Pearson Prentice Hall, Upper Saddle River, NJ.

[3] Rao, S. S., 2011, *Mechanical Vibrations*, 5th ed., Pearson, Upper Saddle River, NJ.

[4] Clough, R. W., and Penzien, J., 2003, *Dynamics of Structures*, 3rd ed., McGraw-Hill, New York.

Appendix

$$\omega_n\left[\frac{rad}{sec}\right] = \sqrt{\frac{3EI}{M_e l^3}}\left[\frac{rad}{sec}\right] = \sqrt{\frac{3(207 * 10^9[Pa])(1.619 * 10^{-11}[m^4])}{(0.23 * 0.140[kg])(0.223[m])^3}} = 167.95\left[\frac{rad}{sec}\right] \quad (11)$$

Table 4: Cantilever beam free vibration trace with *small* displacement.

Time	x_i (Peak amplitude)	T_d	δ (log decrement)	ξ (damping ratio)	ω_n (natural frequency)
(sec)	(Volts)	(sec)	-	-	(rad/s)
t_1 =0.998	x_1 =3.2318	T_{d1} =0.068	0.0420	0.0067	92.40
t_2 =1.066	x_2 =3.0988	T_{d2} =0.066	0.0010	0.0002	95.20
t_3 =1.132	x_3 =3.0957	T_{d3} =0.067	0.0363	0.0058	93.78
t_4 =1.199	x_4 =2.9854	T_{d4} =0.067	-0.0173	-0.0028	93.78
t_5 =1.266	x_5 =3.0376	T_{d5} =0.067	0.0478	0.0076	93.78
t_6 =1.333	x_6 =2.8957	-	-	-	-
	Mean:	0.0670	0.0193	0.0035	93.79
	Std deviation:	0.00071	0.0102	0.0045	0.99

Table 5: Cantilever beam free vibration trace with *large* displacement.

Time	x_i (Peak amplitude)	T_d	δ (log decrement)	ξ (damping ratio)	ω_n (natural frequency)
(sec)	(Volts)	(sec)	-	-	(rad/s)
t_1 =0.556	x_1 =18.2464	T_{d1} =0.066	0.0097	0.0015	95.20
t_2 =0.622	x_2 =18.0709	T_{d2} =0.067	0.0354	0.0056	93.78
t_3 =0.689	x_3 =17.4425	T_{d3} =0.067	0.0231	0.0037	93.78
t_4 =0.756	x_4 =17.0434	T_{d4} =0.068	0.0142	0.0023	92.40
t_5 =0.824	x_5 =16.8022	T_{d5} =0.067	0.0142	0.0023	93.78
t_6 =0.891	x_6 =16.5645	-	-	-	-
	Mean:	0.0670	0.0193	0.0031	93.79
	Std deviation:	0.00071	0.0102	0.0016	0.99

Table 6: Summary of found values.

Value	Units	$Small$	$Large$
ω_n	rad/sec	93.8	93.8
$\delta\omega_n$	rad/sec	1.2	1.2
ξ	n/a	0.0035	0.0031
$\delta\xi$	n/a	0.0056	0.0020
ω_d	rad/sec	93.8	93.8
$\delta\omega_d$	rad/sec	1.2	1.2