Computational Assignment 2 - Kinematics of Crankshaft and Rod

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EXECUTIVE SUMMARY

The objective is to derive and analyze the displacement, velocity, and acceleration of the slider (point B) and the center of mass of the connecting rod (point G) as functions of the crank angle (θ). Using a constant crank angular velocity of 300 rpm, kinematic equations for point B and point G were derived using the geometric method to describe the motion of the system. The analysis was performed for three different crank-to-rod length ratios: P = 0.1, 0.3, 0.5. Computational simulations generated time-based plots of position, velocity, and acceleration for both point B and point G over two full revolutions of the crank. The results show significant variation in position, velocity, and acceleration of points B and G depending on the value of P. As P increases, the magnitude and variation of the slider's velocity and acceleration increase, reflecting stronger dynamic effects. This analysis confirms the importance of geometric ratios in crank-slider mechanisms and supports using Newton's Second Law to identify force trends acting on the system components.

METHOD

To obtain kinematic equations for point B, a geometric approach was used. It consisted of adding the horizontal lengths of the members and setting that equal to the i vector position of B (x_B) at any given time. The vertical distance with respect to B (y_B) was set to 0 as there is no vertical translation at that point. Nevertheless, a kinematic constraint was determined by setting the bar heights equal to each other. The results are the **Equations (1) and (2)** for position below. Taking the derivative of these equations results in the equations for velocity and acceleration - **Equations 3, 4, 5, and 6**. The position, velocity, and acceleration of point G were then evaluated using vector properties based on the given position of G (center of rod l_2).

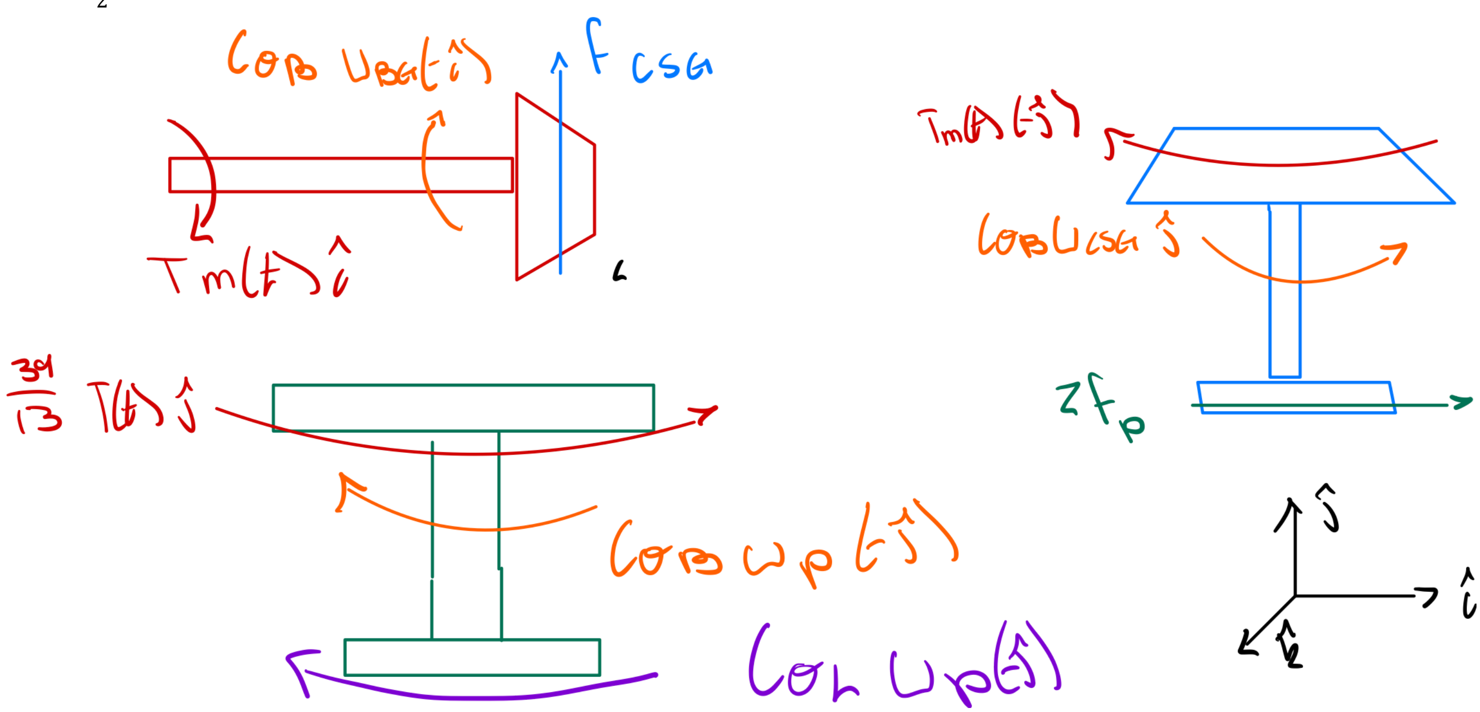

Figure 1: Diagram of Bodies

Geometrical derivation of equations of motion for point B:

Given:

$P = \frac{l_1}{l_2}$, θ'=ω=300 rpm, $l_T = 7ft$, $l_T = l_1 + l_2$

Position (X, Y Components)

$$l_1 cos(\theta) + l_2 cos(\phi) = x_B \quad (1)$$

$$l_1 sin(\theta) = l_2 sin(\phi) \quad (2)$$

Velocity (X, Y Components)

$$-l_1 sin(\theta) * \theta' = l_2 sin(\phi) * \phi' + \dot{x}_B \quad (3)$$

$$l_1 cos(\theta) * \theta' = l_2 cos(\phi) * \phi' \quad (4)$$

Acceleration (X, Y Components)

$$-l_1 sin(\theta) * \theta'' - l_1 cos(\theta) * \theta'^2 - l_2 cos(\phi) * \phi'^2 = l_2 sin(\phi) * \phi'' + \ddot{x}_B \quad (5)$$

$$l_2 cos(\phi) * \phi'' = l_2 sin(\phi) * \phi'^2 + l_1 cos(\theta) * \theta'' - l_1 sin(\theta) * \theta'^2 \quad (6)$$

θ'' term goes to 0 due to constant velocity, zero acceleration condition given

The vector equation properties to find the position, velocity, and acceleration of point G from the EOM derived for point B:

$$r_{OG} = r_{OB} + r_{BG} = x_B(i) + \frac{l_2}{2} * (cos(\phi)(-i) + sin(\phi)j) \quad (7)$$

$$v_G = v_B + \omega_{BG} \times r_{BG} = \dot{x}_B(-i) + \phi'(-k) \times \frac{l_2}{2} * (cos(\phi)(-i) + sin(\phi)j) \quad (8)$$

$$a_G = a_B + \omega_{BG}' \times r_{BG} + \omega_{BG} \times \omega_{BG} \times r_{BG}$$

$$a_G = \ddot{x}_B(-i) + \phi''(-k) \times (\frac{l_2}{2} * (cos(\phi)(-i) + sin(\phi)j)) + \phi'^2(\frac{l_2}{2})(cos(\phi)(i) - sin(\phi)j) \quad (9)$$

Cramer's Rule

To analytically solve for the variables in the equations above, Cramer's Rule was used. Below is an example using the angular velocities.

$$\begin{bmatrix} -l_1 sin(\theta) & -l_2 sin(\phi) \\ l_1 cos(\theta) & l_2 cos(\phi) \end{bmatrix} \begin{bmatrix} \theta' \\ \phi' \end{bmatrix} = \begin{bmatrix} \dot{x}_B \\ 0 \end{bmatrix}$$

$$\theta' = \begin{vmatrix} \dot{x}_B & -l_2 sin\phi \\ 0 & l_2 cos\phi \end{vmatrix} \div \begin{vmatrix} -l_1 sin(\theta) & -l_2 sin(\phi) \\ l_1 cos(\theta) & l_2 cos(\phi) \end{vmatrix}$$

$$\phi' = \begin{vmatrix} -l_1 sin(\theta) & \dot{x}_B \\ l_1 cos(\theta) & 0 \end{vmatrix} \div \begin{vmatrix} -l_1 sin(\theta) & -l_2 sin(\phi) \\ l_1 cos(\theta) & l_2 cos(\phi) \end{vmatrix}$$

After finding a general solution for position, velocity, and acceleration about points B and G, Python was used to create a data set spanning 2 periods using time as the x-axis and the solved terms in the y (seen in the figures below).

PROCEDURE

Using the derived kinematic equations (**Equations 1-6**) for point B, the numpy library in Python was used to construct the model and plot the results. The plots were constructed in terms of P, which is the ratio between l_1 and l_2, to observe the effect of varying the ratio between l_1 and l_2. These figures were compiled to create three subplots in **Figure 2**.

A similar approach was taken for the kinematic equations of point G (**Equations 7-9**) in **Figures 3, 4, 5, 6, and 7**. To ensure no mistakes were made, the kinematic equations of point A were derived geometrically and averaged with B to determine an alternate expression of G (**Appendix Equations**). Since the AB bar is symmetric, the center of mass must be equally positioned between points A and B, rendering this alternate version acceptable for the simulation. In other words, the X position of G is the average of A and B, and the Y position of point G is found by dividing the Y position of A by two. Since this latter approach was significantly easier to code, it was the approach used in the final simulation.

Once these equations and parameters were set, the given constant velocity of θ' (=300 rpm) and θ'' (=0) were used to solve for the rest of the unknowns using the built-in numpy solver, ultimately rendering all the tables and figures discussed.

RESULTS and DISCUSSION

Figure 2 shows the horizontal position, velocity, and acceleration of point B with varying ratios between l_1 and l_2 (P). When P equals 0.1, for example, l_1 is 10% of l_2. The length of l_1 plus l_2 is given as 7ft, therefore when P=0.1, l_1=0.7ft and l_2 =6.3ft. When point A moves below y=0 in the positive x-axis, point A and point B begin moving in the -x direction, until link OA and link AB are both horizontal and lying along the x-axis at y=0. At this position, the horizontal location of points B and A will be at their minimum. Point A will be at x = -0.7ft and point B will be at x = 6.3 ft. **Figure 2** displays this expected result when the position of B is at its minimum, at 6.3ft. The same logic applies to all values of P. Furthermore, the graph displays three points at which the horizontal location of B is 7ft. At these points, links OA and AB are both horizontal along the x-axis, and the horizontal location of A is positive. The first time this occurs is when θ and ϕ are zero. Later occurrences occur at the end of each rotation until reaching the max angle simulated (2π radians).

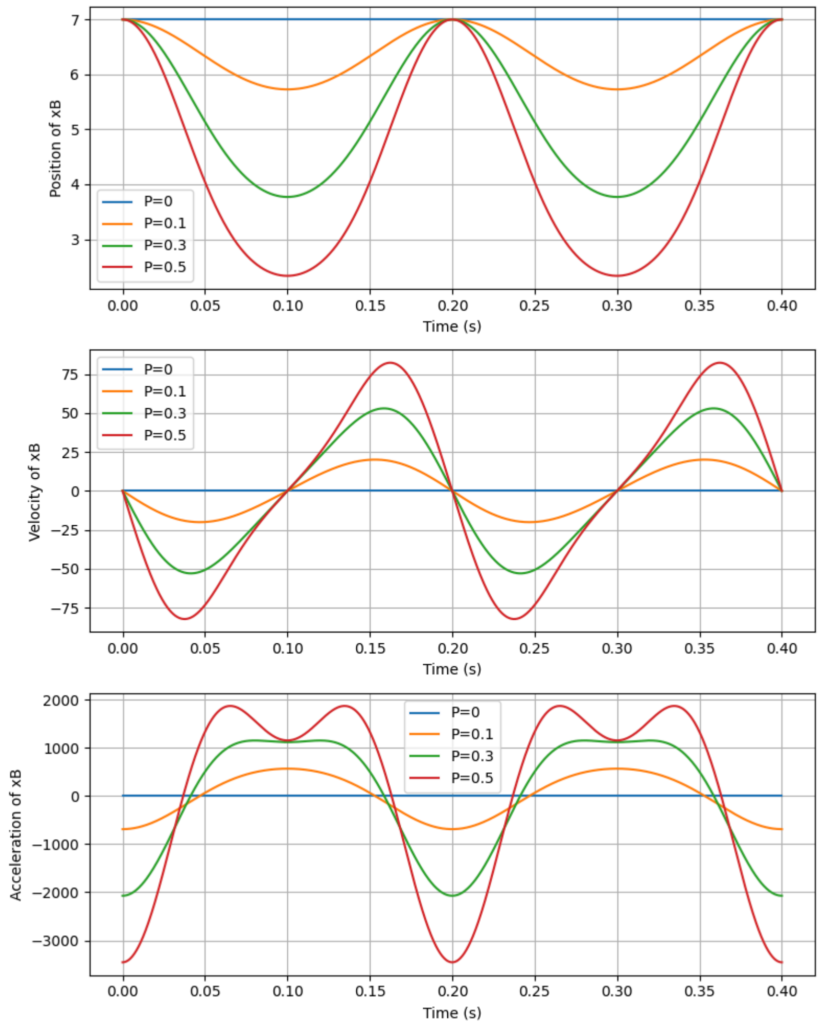

Figure 2: Plot of point B horizontal position, velocity, and acceleration as a function of time with varying P values.

Additionally, the maximum and minimum velocities of point B and the time they occur are displayed in **Figure 2** and recorded in **Table 1** in the appendix. The maximum and minimum acceleration for point B and the time they occur are recorded in **Table 2** in the appendix. The maximum/minimum velocity magnitude is approximately 82.3 ft/s. The magnitude of the minimum acceleration is 3454 ft/s^2, and the magnitude of the maximum acceleration is approximately 1150 ft/s^2. Both the maximum and minimum velocities and acceleration occur when P =0.5, right before and after point A is on the y-axis. The positive maximum acceleration (expanding crankshaft) and by extension the maximum force takes place right before and after the two bars are horizontal to each other, with point A being in the negative x direction. These accelerations also result in accompanying maximum magnitudes of velocity, which come about from these abrupt acceleration local maxima. In the inverse case, where point A is located in the positive x-axis at y=0, the acceleration is at its absolute maximum with the rod fully extended and preparing to contract/expand fully, depending on the cycle direction.

Figure 3 displays the varying vertical position of point G and the corresponding horizontal position. Point G's motion is a combination of a circular motion from the crank rotation, θ, and the back and forth translations of point B. Due to the combined effects, point G's motion is oblong/"egg-shaped", becoming thicker at the bottom of the stroke than the top. This asymmetrical motion begins to appear differently as L_1/L_2 decreases, looking more like a symmetrical oval. This is because when L_1 is small compared to L_2, the motion of L_2 acts more like a linear translation, as the circular motion is less prevalent. This leads to the motion of G being thinner and more symmetrical.

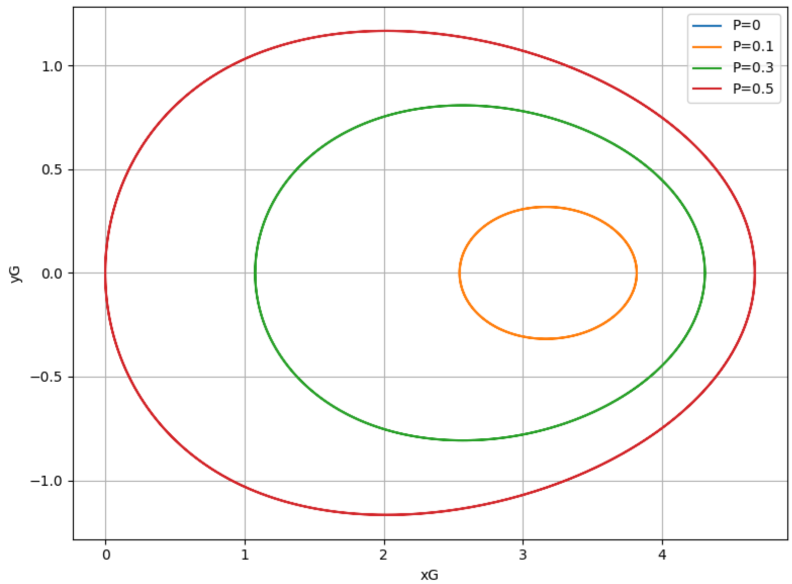

Figure 3: Plot of the motion of point G vertical position vs horizontal position (Y_G vs X_G).

For the P=0 case, there is only one fixed geometry solution available. Therefore, it does not show on the graph as there is no variation in position observable. This same logic applies to all other figures for the position, the velocity, and acceleration of points B and G. Since there is no change in geometry due to the constraints, there is no motion and no observable changes, creating a flat slope at 0.

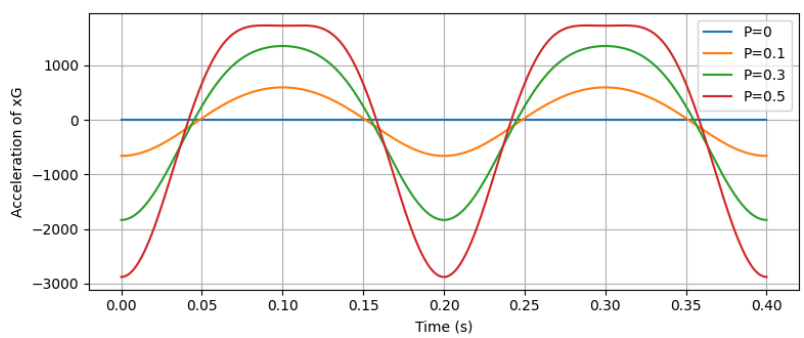

Figure 4: Plot of point G horizontal acceleration as a function of time with varying P values.

Newton's second law can be applied to the problem based on the acceleration. Since force is mass times acceleration, point G and point B experience the greatest force in the x direction, where the magnitude of the acceleration is the greatest, at times indicated by the solution. **Figures 4 and 5** indicate that the acceleration of point G in the horizontal direction has the greatest magnitude where the links reach their maximum or minimum x position (see **Figure 6** in the appendix for the plot of the position and velocity of point G). Therefore, link AB experiences the greatest force at its center of gravity when it changes directions along the x-axis.

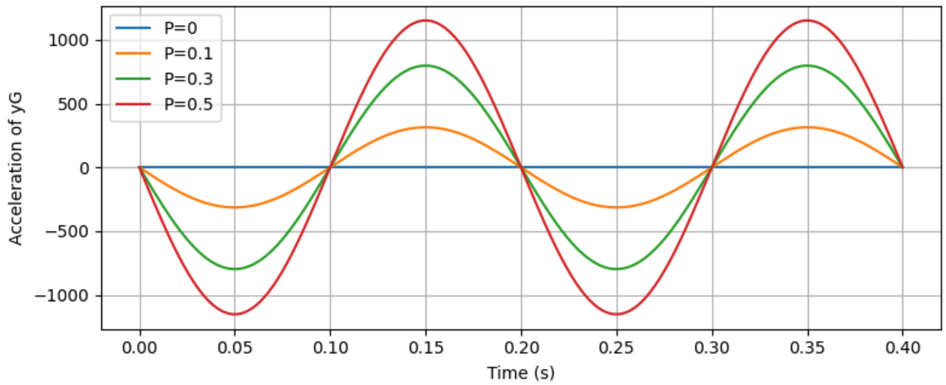

Figure 5: Plot of point G vertical acceleration as a function of time with varying P values.

Figure 5 illustrates the vertical acceleration of point G over time for varying P values. The vertical position in **Figure 7** in the appendix exhibits a sinusoidal pattern that reflects the crank's rotational motion, while the velocity and acceleration curves show how rapidly this vertical motion changes throughout the stroke. As P increases, the vertical velocity and acceleration both grow in magnitude, indicating a stronger vertical influence from the crank's rotation. The peaks in acceleration occur near the midpoints of each stroke, where the vertical direction reverses and inertial forces peak. This vertical dynamic is particularly important in applications like internal combustion engines, where vertical forces translate into vibrations and stresses that impact structural design and operational smoothness.

CONCLUSIONS

In conclusion, the kinematic analysis of the crank-slider mechanism reveals significant dependence on the crank-to-rod length ratio (P). As P increases, the dynamic effects on both point B and point G become more pronounced, with greater velocity and acceleration magnitudes. These findings have direct implications for the design and performance of reciprocating systems, where balancing force transmission and inertial effects is crucial. Newton's Second Law reinforces that maximum force corresponds to maximum acceleration, which occurs at predictable angular positions (fully expanded or contracted). This analysis provides valuable insight into optimizing component dimensions to reduce wear, minimize vibrations, and ensure efficient mechanical performance. Ultimately, this makes the mechanism ideal for the use case given of the crankshaft two-stroke engine.

APPENDIX

Point B Maximum and Minimum Velocities	
Time (s)	**Velocity (ft/s)**
0.0376	-82.3
0.163	82.3
0.237	-82.3
0.362	82.3

Table 1: Maximum and minimum velocity of point B and the time it occurs.

Point B Maximum and Minimum Acceleration	
Time (s)	**Acceleration (ft/s^2)**
0	-3454
0.119	1150
0.080	1150
0.200	-3454
.280	1150
0.319	1150
0.400	-3454

Table 2: Maximum and minimum acceleration of point B and the time they occur.

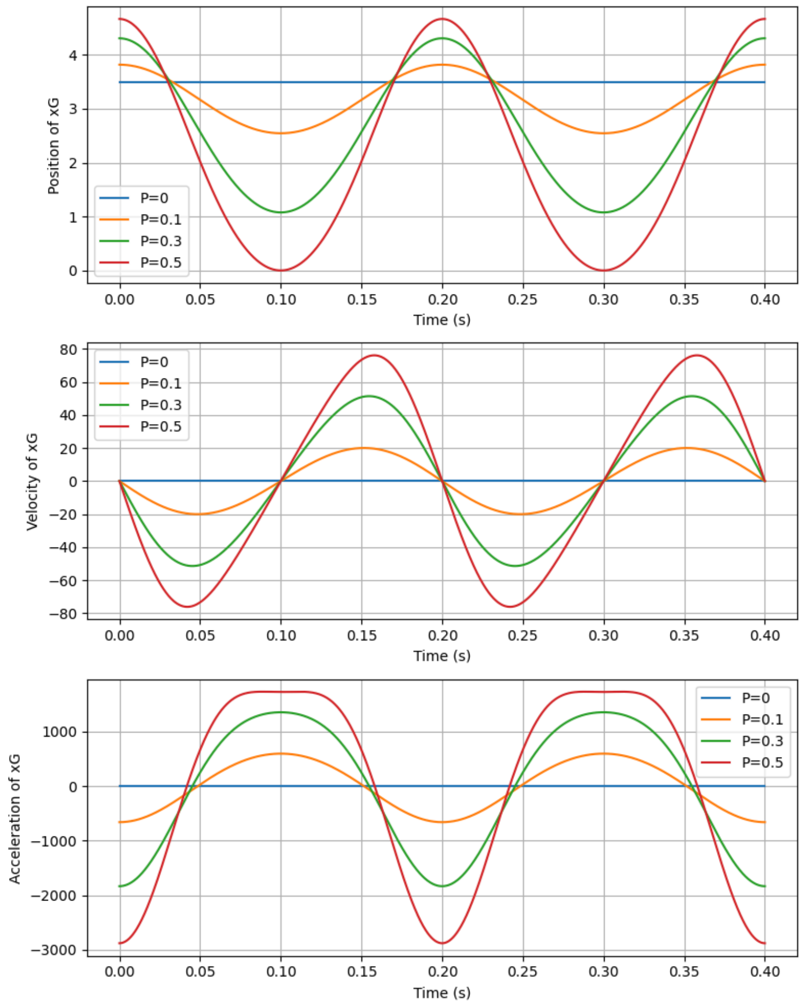

Figure 6: Plot of point G horizontal acceleration as a function of time with varying P values.

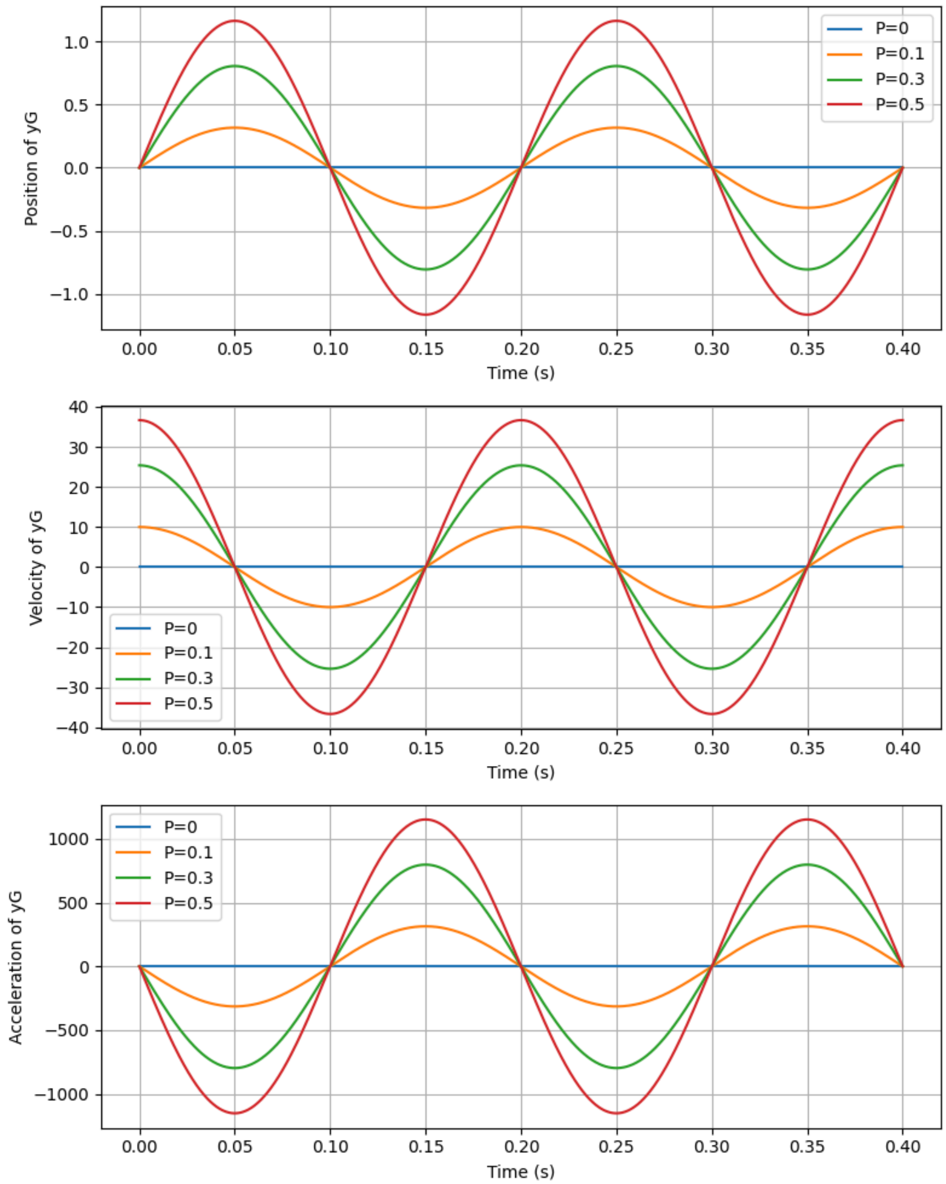

Figure 7: Plot of point G vertical position, velocity, and acceleration as a function of time with varying P values.

Alternative proof for point G, used to verify equations, and the simulation used:
For point G, an average between points A and B was taken:

Point A:
Position (X, Y Components)

$$l_1 cos(\theta) = x_A \tag{A.1}$$

$$l_1 sin(\theta) = y_A \tag{A.2}$$

Velocity (X, Y Components)

$$-l_1 sin(\theta) * \theta' = \dot{x}_A \tag{A.3}$$

$$l_1 cos(\theta) * \theta' = \dot{y}_A \tag{A.4}$$

Acceleration (X, Y Components)

$$-l_1 sin(\theta) * \theta'' \; - l_1 cos(\theta) * \theta'^2 = \ddot{x}_A \quad (A.5)$$

$$l_1 cos(\theta) * \theta'' - l_1 sin(\theta) * \theta'^2 = \ddot{y}_A \quad (A.5)$$

Average of points A and B, for point G:

Position

$$\frac{x_A + x_B}{2} = x_G = (l_1 cos(\theta) + \frac{1}{2} l_2 cos(\phi)) \quad (A.6)$$

$$\frac{y_A}{2} = y_G = \frac{1}{2} * l_1 sin(\theta) \quad (A.7)$$

Velocity

$$\frac{\dot{x}_A + \dot{x}_B}{2} = \dot{x}_G \quad (A.8)$$

$$\frac{\dot{y}_A}{2} = \dot{y}_G = \frac{1}{2} * l_1 cos(\theta) * \omega \quad (A.9)$$

Acceleration

$$\frac{\ddot{x}_A + \ddot{x}_B}{2} = \ddot{x}_G \quad (A.10)$$

$$\frac{\ddot{y}_A}{2} = \ddot{y}_G = \frac{1}{2} (l_1 cos(\theta) * \theta'' - l_1 sin(\theta) * \omega^2) \quad (A.11)$$

θ'' term goes to 0 due to constant velocity, zero acceleration condition given

REFERENCES

[1] Dynamics in Engineering Practice, 11th Ed (only), D.W. Childs & Ap. Conkey, CRC Pubs https://www.crcpress.com/Dynamics-in-Engineering-Practice-Eleventh-Edition/Childs-Conkey/9781482250251