Computational Assignment 3 - Modeling a Mechanical System

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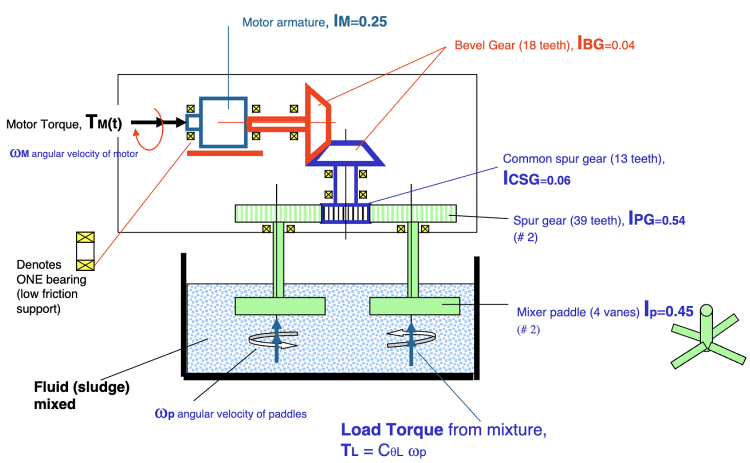
EXECUTIVE SUMMARY

The objective is to derive equations of motion and find the total energy and dissipative power of the system. The equation of motion can be found through two methods. First, an equation of motion for the system is established by summing forces and moments for the five components and implementing the kinematic constraints of the system. The second method for finding the equation of motion is by analyzing the potential, kinetic, and dissipated energy of the system and using the concept of conservation of mechanical energy. Additionally, given specific damping coefficients and the motor speed, the motor torque and paddle speed are calculated. They will require a minimum of 323 newton meters of torque to drive the system, and will take 3.27 seconds from beginning at rest to reach 99% of operating speed. Due to power losses in the system, such as in the bearings, the efficiency of the system is 84%, and it costs $438 per day, $75 of which was lost due to friction.

METHOD

The mixer system presented had many moving parts. Under further inspection, they can be broken up into five distinct free bodies:



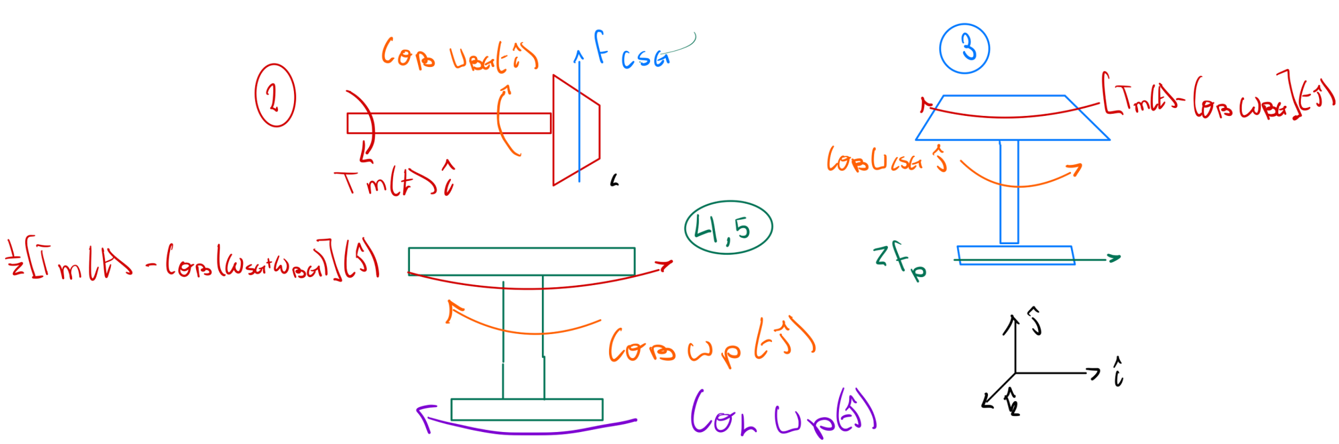


Figure 1: Free Body Diagrams for Components

Using this approach, the system can be described using 5 equations - one for each of the bodies:

$$I_M\theta_1'' = T_M - T_{12} - \theta_1'C_{\theta B} \tag{1}$$

$$I_{BG}\theta_2'' = T_{12} - T_{23} - 2\theta_2'C_{\theta B} \tag{2}$$

$$I_{BG}\theta_3'' = T_{23} - T_{34} - 2\theta_3'C_{\theta B} \tag{3}$$

$$I_{PG}\theta_4'' = T_{34} - T_{45} \tag{4}$$

$$I_P\theta_5'' = T_{45} - T_L = T_{45} - C_{\theta L}\omega_P - \theta_5'C_{\theta B} \tag{5}$$

Furthermore, some kinematic constraints are used to simplify these equations. The defined gear ratios and relationships between the bodies were used to explain how the motor's input torque is converted into output torque at the paddles.

$$\theta_1 = \theta_2 = \theta_3 = \theta_M \tag{6}$$

$$\theta_4 = \theta_5 = \theta_P \tag{7}$$

$$3\theta_M = \theta_P, 3\theta_M' = \theta_P', 3\theta_M'' = \theta_P'' \tag{8}$$

The five motion equations can be simplified to two, where θ_M is the angle of the motor, θ_P is the angle of the paddle, T_{CSG} is the torque of the center spur gear, T_{SG} is the torque of the additional spur gears, and the relationship of the torques are applied:

$$\theta_M''[I_M + 2I_{BG} + I_{CSG}] = T_M - 5\theta_M'C_{\theta B} - T_{CSG} \tag{9}$$

$$T_{CSG} = \frac{13}{39}T_{SG} \tag{10}$$

$$\theta_P''[2(I_{PG} + I_P)] = T_{SG} - 2C_{\theta B}\theta_P' - 2C_{\theta L}\omega_P \tag{11}$$

After this step, the equations are simplified even further, into a single equation.

$$\Rightarrow 9\theta_P''[I_M + 2I_{BG} + I_{CSG}] + \theta_P''[2(I_{PG} + I_P)] = 3T_M - 15\theta_P'C_{\theta B} - 2C_{\theta B}\theta_P' - 2C_{\theta L}\omega_P$$

$$\Rightarrow [9(I_M + 2I_{BG} + I_{CSG}) + 2(I_{PG} + I_P)]\theta_P'' = 3T_M + [-47C_{\theta B} - 2C_{\theta L}]\theta_P'$$

$$\Rightarrow 3T_M = \theta_P''[9(I_M + 2I_{BG} + I_{CSG}) + 2(I_{PG} + I_P)] + [47C_{\theta B} + 2C_{\theta L}]\theta_P' \tag{12}$$

Therefore,

$$I_{eq} = 9(I_M + 2I_{BG} + I_{CSG}) + 2(I_{PG} + I_P) \quad (13)$$

$$C_{eq} = 47C_{\theta B} + 2C_{\theta L} \quad (14)$$

Using the conservation of mechanical energy equation, P_{diss} (dissipative power) and P_{drive} (external drive power) can be used to describe the non-conservative forces in the system (from the damping and driving torque):

$\frac{d}{dt}(E_k + E_p) + P_{diss} = P_{drive}$, where E_k is the kinetic energy and E_p is the potential energy. (15)

$$\frac{d}{dt}E_k = [9(I_M + 2I_{BG} + I_{CSG}) + 2(I_P + I_{PG})]\theta_P'' * \theta_P' \quad (16)$$

$$P_{diss} = C_{\theta B}(\frac{d}{dt}\theta_1)^2 + 2C_{\theta B}(\frac{d}{dt}\theta_2)^2 + 2C_{\theta B}(\frac{d}{dt}\theta_3)^2 + 2C_{\theta B}(\frac{d}{dt}\theta_5)^2 + 2C_{\theta L}(\frac{d}{dt}\theta_5)^2 \quad (17)$$

$$P_{diss} = [47C_{\theta B} + 2C_{\theta L}](\frac{d}{dt}\theta_P)^2 = [47C_{\theta B} + 2C_{\theta L}](\theta_P') * (\theta_P') \quad (18)$$

$$P_{drive} = (T_M) * 3\theta_P' \quad (19)$$

The simplified energy equation is as follows, using (16), (18), and (19):

$$[9(I_M + 2I_{BG} + I_{CSG}) + 2(I_P + I_{PG})]\theta_P'' + [47C_{\theta B} + 2C_{\theta L}](\theta_P') = 3(T_M) \quad (20)$$

Therefore, the same I_{eq} and C_{eq} were determined as in the Newtonian approach:

$$I_{eq} = 9(I_M + 2I_{BG} + I_{CSG}) + 2(I_{PG} + I_P) \quad (13)$$

$$C_{eq} = 47C_{\theta B} + 2C_{\theta L} \quad (14)$$

PROCEDURE

Using the results obtained from the kinematic constraints, free body diagrams, and equations of motion, the mixer can be analyzed under steady state and transient conditions. Using the values that were provided, this is the current state of the system:

- Motor torque T_M for a <u>constant</u> motor speed of 60 Hz (377 rad/s):
  - $T_M * 3 = \theta_P''[9(I_M + 2I_{BG} + I_{CSG}) + 2(I_{PG} + I_P)] + [47C_{\theta B} + 2C_{\theta L}]\theta_P'$
  - $= 0 + [47(0.028) + 2(3.2)](125.66)$
  - T_M = 323.2 Nm

- Power from the motor:
  - $P = T * \omega$
  - $P = 323.2 * 377$
  - $P = 121.8$ kW
- Power lost from bearings:
  - $\omega_{CSG} = \omega_M = 377\, rad/s$
  - $\omega_P = \omega_M * \frac{N_{CSG}}{N_{PG}} = 377 * \frac{1}{3} = 126\, rad/s$
  - $P_{diss} = [47C_{\theta B}](\theta_P')^2 = [47(0.028)](126)^2$
  - Total Power Lost, $P = 20.8kW$
- Power from the load:
  - $P_L = T_L * \omega = C_{\theta L} * \omega^2$
  - $P_L = 2 * 3.2 * 126^2$
  - $P_L = 101.1kW$
- Mechanical efficiency:
  - $\eta = \frac{P_L}{P_M} * 100 = \frac{102}{122} * 100$
  - $\eta = 84\%$
- Cost to operate:
  - $C = 0.15 \frac{\$}{kW*h}$
  - $Energy = Power * t = 121.8 * 24 = 2923\, kW * h$
  - $Cost\ Per\ Day = E * C$
  - $Cost\ Per\ Day = \$438$
- Money lost:
  - $Energy\ Lost = Bearing\ Power * t = 20.8 * 24 = 499\, kW * h$
  - $Cost\ Per\ Day = Energy\ Lost * C$
  - $Money\ Lost = \$75$
- System time constant:
  - $I_{eq} = 9(I_M + 2I_{BG} + I_{CSG}) + 2(I_{PG} + I_P)$
  - $= 9(0.25 + 2(0.04) + 0.06) + 2(0.54 + 0.45)$
  - $= 5.49\, kg * m^2$
  - $C_{eq} = 17C_{\theta B} + 2C_{\theta L} = 47(0.028) + 2(3.2) = 7.72 \frac{Nm}{rad/s}$
  - $\tau = \frac{I_{eq}}{C_{eq}} = \frac{5.49}{7.72}$
  - $\tau = 0.711\, s$
- Time Response:
  - $\omega_p = \omega_{steady}(1 - e^{-t/\tau})$
  - $\omega_p = 126(1 - e^{-t/0.711})$
- Time to reach operating speed:
  - $124.74 = 126(1 - e^{-t/0.711})$
    - 124.74 is 99% of the operating speed. This value is used because ln(0) does not exist.

- $0.99 = 1 - e^{-t/0.711}$
- $ln(0.01) =- t/0.711$
- $t = 3.27s$

RESULTS and DISCUSSION

The gear ratios are a significant part of the system and are used to simplify the equations. By applying the gear ratio kinematic constraints and using the Newtonian method, the system is modeled in **Equation 12**. The equations of motion are found by summing forces and torques between the gears, as indicated by the free-body diagrams in **Figure 1**. The system's equivalent inertia and viscous damping coefficient are noted in **Equations 13** and **14**.

Additionally, the energy method is applied to the system to verify the first approach. The two main components in the energy equations are the power lost from the bearings and load, the torque from the motor, and the power due to kinetic energy. The energy method also uses the gear ratio kinematic constraints to simplify the result, and the final equation (**Equation 20**) results in the same equation of motion as the Newtonian method, as expected. The system's equivalent inertia and viscous damping coefficients are equivalent to those calculated using the Newtonian method, which verifies the results of both methods.

Considering the given values, several key quantities from the system are calculated in the procedure, and recorded in **Table 1**. It is important to note that the power in the motor can be calculated from the torque of the system and verified by summing both the dissipated power and the power of the load.

Table 1: Key values from the system.

T_{Motor}	323Nm
P_{Motor}	121.8kW
P_{diss}	20.8kW
P_{Load}	101.1kW
η	84%
$Cost\ Per\ Day$	\$438
$Money\ Lost$	\$75
τ	0.711s
$Time\ to\ reach\ operating\ speed$	3.27s

The money lost is due to the power losses in the system. To reduce the money lost each day, the frictional losses in the bearings should be reduced by implementing a maintenance routine on the system and applying lubricant to ensure smoother bearings. Furthermore, the paddles could be designed to be more efficient. If the losses are reduced, the efficiency of the system would be improved.

$$126(1 - e^{-t/0.724}) \tag{21}$$

Equation 21 is the equation for the mixer speed as a function of time. **Figure 2** displays a graphical representation of the speed. The time constant, τ, is the value at which the system reaches 63% of its operational speed. **Figure 2** visually confirms the time constant of 0.724 seconds, showing that the speed reaches approximately 63% of its steady-state value. By observation of **Figure 2**, the time at which the system reaches its operating speed is around 3 seconds, validating prior calculations recorded in **Table 1**.

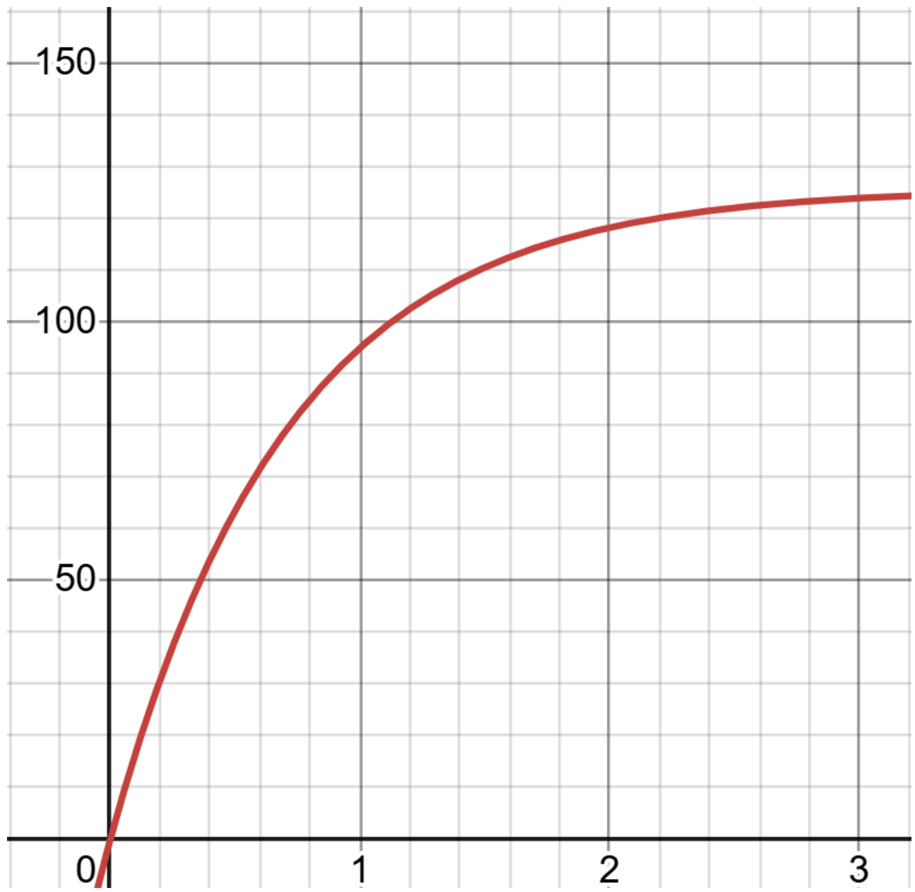


Figure 2. Mixer Speed (rad/s) as a Function of Time (s)

CONCLUSIONS

The mechanical system for the mixer was modeled successfully by applying principles of dynamics and energy conservation. The equivalent inertia and viscous damping parameters were determined consistently through independent approaches. Steady-state performance calculations established the required motor torque, operating speeds, and mechanical power losses within the system. The startup response analysis determined the system time constant and characterized the transient behavior of the paddles. As such, due to its mode of operation, this mixer can probably be classified as a “Paddle industrial mixer” [2]. To further improve its performance, an optimal angle can be set for the viscous fluid being mixed. From certain sources, it can be determined that a 45-degree angle [3] would provide the least amount of resistive torque from the fluid. The modeling effort clearly demonstrates the importance of accounting for mechanical inefficiencies, especially bearing and load damping, to accurately predict system performance. Overall, the work validates the modeling approach and highlights the need for precise dynamic characterization in the design and operation of mechanical systems.

APPENDIX

Detailed steps regarding the derivation of the EOM of the system using the power equation:

From (15) to (16), using the kinematic constraints to simplify:

$$\frac{d}{dt}E_k + 0 = (9I_M\theta_M'' + 2 * 9I_{BG}\theta_M'' + 9I_{CSG}\theta_M'' + 2I_{PG}\theta_P'' + 2I_P\theta_P'') * \theta_P'$$

To determine the left side of (15) in (16):

$$E_k = \frac{1}{2}I_M\theta_1'^2 + \frac{1}{2}I_{BG}\theta_2'^2 + \frac{1}{2}(I_{BG} + I_{CSG})\theta_3'^2 + 2\frac{1}{2}I_{PG}\theta_4'^2 + 2\frac{1}{2}I_P\theta_5'^2$$

$$E_p = 0$$

Using the kinematic constraints to simplify (17) to (18):

$$P_{diss} = C_{\theta B}(\frac{d}{dt}\theta_M)^2 + 2C_{\theta B}(\frac{d}{dt}\theta_M)^2 + 2C_{\theta B}(\frac{d}{dt}\theta_M)^2 + 2C_{\theta B}(\frac{d}{dt}\theta_P)^2 + 2C_{\theta L}(\frac{d}{dt}\theta_P)^2$$

$$P_{diss} = C_{\theta B}9(\frac{d}{dt}\theta_P)^2 + 2C_{\theta B}9(\frac{d}{dt}\theta_P)^2 + 2C_{\theta B}9(\frac{d}{dt}\theta_P)^2 + 2C_{\theta B}(\frac{d}{dt}\theta_P)^2 + 2C_{\theta L}(\frac{d}{dt}\theta_P)^2$$

To find the final EOM using energy in (20) from the previous equations:

$$\frac{d}{dt}(E_k + E_p) + P_{diss} = P_{drive}$$

$$[9(I_M + 2I_{BG} + I_{CSG}) + 2(I_P + I_{PG})]\theta_P'' * (\theta_P') + [47C_{\theta B} + 2C_{\theta L}](\theta_P') * (\theta_P') = (T_M) * \theta_M'$$

$$[9(I_M + 2I_{BG} + I_{CSG}) + 2(I_P + I_{PG})]\theta_P'' * (\theta_P') + [47C_{\theta B} + 2C_{\theta L}](\theta_P') * (\theta_P') = \frac{1}{3}(T_M) * \theta_p'$$

(Notice how the θ_P'/ω values cancel out to get a familiar equation of motion.)

REFERENCES

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